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Orbital Restriction on the Functional Form of the Lagrangian

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Abstract

The fact that a non-relativistic particle describes an orbit imposes a restriction on the functional form of the Lagrangian. For a classical particle subjected to an arbitrary local 'generalized force,' the (local) Lagrangian is shown to involve, at highest, first, second and third time-derivatives of the position respectively, in one, two and three dimensions. The generalization in the quantum régime is indicated with the aid of Feynman's path integrals.

1. *Introduction*

The derivation of Lagrange's equation of motion from Hamilton's principle has been often described as 'elegant' (Goldstein, 1950). Besides, this approach to dynamics has the further merit that the Lagrangian to be constructed involves only physical quantities such as kinetic and potential energy which characterize the motion of the system. Thus, this formulation is automatically invariant with respect to the choice of coordinates for the system. Another advantage is that the Lagrangian formulation can be extended easily to describe systems which are not normally considered in dynamics such as the electromagnetic fields, quantum fields of elementary particles, and so on (Wentzel, 1949). The fact that the Lagrangian in general involves time-derivatives of lower orders than the equation of motion, Lagrange's equation, is a further advantage. This is so in particular when one is looking for a *new* equation of motion since one may start dealing with quantities of lesser conceptual complexities.

Ordinarily, the Lagrangian for a system is chosen such that the equation of motion derived from it should be in agreement with physical experience, i.e. should be successfully tested with experiments. In most cases, the Lagrangian L has been chosen to be a functional of the position and its first time-derivative, velocity, because the empirically tested equation of motion, Newton's equation, etc., can be derived from this form of L. However, if one were unaware of Newton's law beforehand or, more generally, unavailable of definite laws of motion, and were looking for new equations of motion, one would have started with the Lagrangian involving the derivatives of still higher orders. Some authors considered such general cases and

discussed the mathematical structure of the hypothetical laws of motion involving high derivatives (Barut & Mullen, 1962; Thielheim, 1967). An interesting question may be raised whether there should exist any reasonable upper bounds on the order of the derivatives which enter into the Lagrangian and hence into the equation of motion. The present paper concerns this question and it is pointed out, and shown, that for a non-relativistic particle in an arbitrary '*local* force field' the Lagrangian should involve, at highest, first, second and third time-derivatives respectively, in one, two and three dimensions. This restriction arises from the fact that particle describes an orbit. This restriction has the physical meaning also in the quantum régime, where the action is measured in units of Planck's constant $\hbar$ if the quantization is prescribed according to Feynman's path integrals (Feynman & Hibbs, 1965).

2. *Classical Motion in Free Space*

Let us consider a mass point, hereafter called a particle, in the three-dimensional space. Let us suppose that the particle moves from the space-time point $(\mathbf{r}_1, t_1)$ to $(\mathbf{r}_2, t_2)$. This motion is completely known if the position $\mathbf{r}(t)$ of the particle is found as a function of time t,

$$\mathbf{r} = \mathbf{r}(t), \qquad t_1 \leqq t \leqq t_2 \tag{2.1}$$

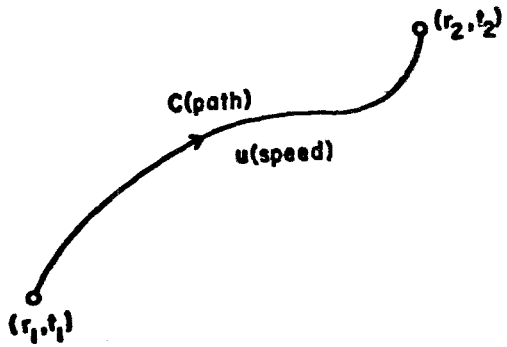

Figure 1—The orbit from $(\mathbf{r}_1, t_1)$ to $(\mathbf{r}_2, t_2)$ can be characterized by the path C and the speed $u(t)$ with which the particle proceeds along C.

An equivalent way of describing the motion is to know the path, i.e. the curved line of passage C and the speed $u(t)$ with which the particle proceeds along C. If the arc length of the curve from $\mathbf{r}_1(t_1)$ is denoted by $s \geqq 0$, then the speed $u(t)$ is given by

$$u(t) \equiv \frac{ds(t)}{dt} > 0 \tag{2.2}$$

Let us now consider an infinitesimal segment of the curve. This segment is characterized by the position and the unit tangent $\mathbf{t} \equiv d\mathbf{r}/ds$. The direction of the tangent $\mathbf{t}$ in three dimension may be specified by *two* angles with respect to a set of fixed axes. By giving these angles as functions of s, one may then characterize the whole curve. However, this sort of specification depends on the choice of coordinates. As it is well known in differential geometry, one can avoid this in the following manner. Introduce the three *local* orthonormal vectors, i.e. tangent $\mathbf{t}$, normal $\mathbf{n}$ and binormal $\mathbf{b}$. They change according to the Fresnet formulae,

$$\begin{aligned} \frac{d\mathbf{t}}{ds} &= \kappa\mathbf{n} \\ \frac{d\mathbf{b}}{ds} &= -\tau\mathbf{n} \\ \frac{d\mathbf{n}}{ds} &= \tau\mathbf{b} - \kappa\mathbf{t} \end{aligned} \tag{2.3}$$

where $\kappa(\geqq 0)$ and τ are numbers, called respectively *curvature* and *torsion*. One can then characterize the whole curve by giving κ and τ as functions of s [the fundamental theorem in differential geometry (Struik, 1950)]. Conversely if a curve is given by $\mathbf{r} = \mathbf{r}(s)$, then the local parameters κ and τ are given by the s-derivatives of $\mathbf{r}$. If Cartesian coordinates are used, $\kappa(s)$ and $\tau(s)$ are given by

$$\begin{aligned} \kappa(s)^2 &= \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 \\ \tau(s) &= \frac{1}{\kappa^2(s)} \begin{vmatrix} \dfrac{dx}{ds} & \dfrac{dy}{ds} & \dfrac{dz}{ds} \\ \dfrac{d^2x}{ds^2} & \dfrac{d^2y}{ds^2} & \dfrac{d^2z}{ds^2} \\ \dfrac{d^3x}{ds^3} & \dfrac{d^3y}{ds^3} & \dfrac{d^3z}{ds^3} \end{vmatrix} \end{aligned} \tag{2.4}$$

It is emphasized that the characterization of the curve requires *two* independent local variables in *any* specifications. That is a characteristic of the space in which the curve runs. If the curve is restricted to run on a *two*-dimensional plane, *one* local parameter would have been sufficient.

Unlike the position vector $\mathbf{r}(t)$ in the equation of orbit (2.1), the speed $u(t)$, the curvature $\kappa(s)$ and the torsion $\tau(s)$ in (2.2–2.4) are *scalars*, and in fact do not depend on the choice of the coordinate system. These scalars may be thus called *intrinsic* local parameters for the motion. Therefore, the Lagrangian, L_1, must be defined through these local parameters, and will be given by a functional of the same,

$$L_1 = L_1[\kappa(s(t))\,\tau(s(t)), u(t)] \tag{2.5}$$

When applying Hamilton's principle to derive the equation of motion, it is customary to express the Lagrangian in terms of a chosen set of coordinates e.g. (x,y,z) and their time-derivatives.

Since

$$\frac{dx(s(t))}{dt} = \frac{dx}{ds}\frac{ds}{dt} = \frac{dx}{ds}u(t) \quad \text{or} \quad \frac{dx}{ds} = \frac{1}{u}\frac{dx}{dt} = \frac{1}{u}\dot{x}$$

$$\frac{d^2x}{ds^2} = \frac{1}{u}\frac{d}{dt}\left(\frac{1}{u}\frac{dx}{dt}\right) = \frac{1}{u^2}\frac{d^2x}{dt^2} - \frac{1}{u^3}\frac{du}{dt}\frac{dx}{dt} \tag{2.6}$$

By this calculation we can infer and show that $\kappa(s(t))$ can be expressed in terms of up to the second time-derivatives of the coordinates, i.e. velocity and acceleration. In a similar manner we can show that the torsion $\tau(s(t))$ can be expressed in terms of up to the third time-derivatives. We thus find that the Lagrangian $L_1\ [\kappa,\tau,u]$ can contain up to third time-derivatives but no higher derivatives. We also see that the highest order of the space derivatives is transferred without change in that of the time-derivatives.

3. *Classical Motion in Anisotropic Space*

The motion of the particle will in general be affected by the surroundings. This environmental influence will depend on the location $\mathbf{r}(t)$ of the particle, or in more general cases will depend on the kinetic state of motion if correlation should exist between the influence and the state of motion. If the influence were to depend only on the location $\mathbf{r}(t)$ and the kinetic characteristics (κ,τ,u), then it will be said to be a *local* influence. Such local influence must be characterized by the Lagrangian

$$L_2 = L_2[\mathbf{r}(t), \kappa\ \tau, u, t] \tag{3.1}$$

If we note that the kinetic parameters can be specified by up to third time-derivatives, then we may write the Lagrangian L_2 as

$$L_2' = L_2'[\mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}, t] \tag{3.2}$$

instead of (3.1).

The total Lagrangian L will be defined as

$$L \equiv L_1 + L_2' \tag{3.3}$$

which can contain up to third time-derivatives but no higher derivatives. This establishes our major assertion stated in the introduction.

Now, the variational principle

$$\delta = \delta \int_{t_1}^{t_2} L(\bar{x},\bar{y},\bar{z},\ddot{x},\ldots,\dot{x},\ldots,x,\ldots,t)\,dt = 0 \tag{3.4}$$

may be applied in a usual manner, except for the fact that the variation is subject to the subsidiary condition

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1 \tag{3.5}$$

which characterizes the space curve.

In general, the variational principle of the form (3.4) could yield the simultaneous equations of sixth order, but none of higher orders. It is noted that the advantage of the Lagrangian formulation is clearly seen here. It is difficult and almost impossible to conceive a general equation of sixth order in time.

If the motion of the particle is restricted to a plane, then the path is characterized by the two-dimensional curvature κ alone, which involves the second derivatives. Then, by a similar argument we can deduce that the Lagrangian may contain up to second time-derivatives, i.e. position, velocity and acceleration. In the one-dimensional motion, the path is specified by the direction along the line axis, i.e. the sign of dx/ds. The Lagrangian will be then a functional of position and speed (and time).

The conclusion obtained here is rather striking. It is notable that the argument is essentially based on the assumed *locality* of the Lagrangian. The argument does not apply to a non-local theory, which is, however, harder to conceive than a local theory.

4. *Examples and Remarks*

In the case when the classical particle is subjected to a conservative force

$$\mathbf{F}(\mathbf{r}) = -\nabla\phi(\mathbf{r}) \tag{4.1}$$

where $\phi(\mathbf{r})$ is the potential energy, the Lagrangian L is chosen to be

$$L = \tfrac{1}{2}Mu^2 - \phi(\mathbf{r}) \tag{4.2}$$

where M is the mass of the particle. This is the choice which gives rise to Newton's equation of motion

$$\frac{Md^2\mathbf{r}}{dt^2} = -\nabla\phi(\mathbf{r}) \tag{4.3}$$

It is notable that the form (4.2) is consistent with the general characterization of the Lagrangians, (2.5) and (3.1). It is interesting to note that the motion prescribed by the traditional classical mechanical rule does not depend on the geometrical parameters (κ, τ). If the motion were to depend on these parameters, the mechanics would have been quite different, depending strongly on the dimensional order of the space.

5. *Quantum Generalization*

The argument with respect to the orbital-motional restriction on the Lagrangian can be carried over in the quantum régime, where the action

$$I \equiv \int_{t_1}^{t_2} L$$

is measured in the unit of Planck's constant $\hbar$. This can be done simply by constructing (new) quantum mechanics with the aid of path integrals

(Feynman & Hibbs, 1965). The probability $P(\mathbf{r}_2 t_2 \mathbf{r}_1 t_5) \equiv P(2,1)$ to go from $(\mathbf{r}_1 t_1)$ to $(\mathbf{r}_2 t_2)$ is by postulate given by the absolute square

$$P(2,1) = |K(2,1)|^2 \tag{5.1}$$

of an amplitude $K(2,1)$ to go from 1 to 2; this amplitude is defined by the sum of the contribution $\Phi[\mathbf{r}(t)]$ from each path.

$$K(2,1) = \sum_{\text{all paths}} \Phi[\mathbf{r}(t)] \tag{5.2}$$

The contribution of a path has a phase proportional to the action $I[\mathbf{r}(t)]$

$$\Phi[\mathbf{r}(t)] = \text{const.} \exp(i/\hbar)\, I[\mathbf{r}(t)] \tag{5.3}$$

where the action is that for the corresponding classical system. The usual Schrödinger-like description in terms of the wave function $\psi(\mathbf{r}, t)$ can be generated from the integral equation

$$\psi(\mathbf{r}_2, t_2) = \int d^3 r_1\, K(\mathbf{r}_2 t_2, \mathbf{r}_1 t_1)\, \psi(\mathbf{r}_2, t_2) \tag{5.4}$$

The further detailed discussion will not be given in the present paper, for this would become meaningful only when a particular form of Lagrangian is specified and used in the discussion.

The generalization of the argument to the case of a relativistic particle and to that of a qunatum field is currently under investigation.

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